Kalman Filters in English*

Yoshi, Nori, & J

September 24, 2002

Goal: To get an idea of what parts of the noise are relevant

k	=	kth iteration of time
$\vec{x}(k)$	=	start state — where we are now, at time k — a triplet vector: $\vec{x}(k) = (x, y, \theta)$
$\vec{x}(k+1)$	=	estimated state based on control inputs
$\vec{z}(k+1)$	=	estimated state based on measurement from sensors
$\vec{u}(k)$	=	control inputs — where our wheels, odometry, &c. told us we went — also of the form (x, y, θ)
$P(k)$	=	current uncertainty — how unsure we are of our present state
I	=	the identity matrix
A, B, B', M	=	matrices which are functions of the environment (ξ)
V, W	=	noise
C_v, C_w	=	covariance matrices

$$\begin{aligned}
\text{Estimated Current State:} \quad \vec{x}(k+1) &= A\vec{x}(k) + B\vec{u}(k) + V \\
\text{Measurement Equation:} \quad \vec{z}(k+1) &= M\vec{x}(k+1) + W \\
\text{Uncertainty Equation:} \quad P'(k+1) &= AP(k)A^T + B'C_vB'^T \\
\text{Uncertainty Update Equation:} \quad P(k+1) &= \underbrace{\left[I - K(k+1)M\right]}_{>[0],<I} \cdot P'(k+1)
\end{aligned}$$

Kalman Gain: $K(k+1) = \left[P'(k+1) \cdot M^T\right] \cdot \left[MP'(k+1)M^T + C_w\right]^{-1}$

$P'(k+1)$, the uncertainty equation, would increase indefinitely if we never got any new information about our surroundings — all we do in it is add the noise, C_v, to our current uncertainty (times $A \cdot A^T$ to account for conditions). But we are getting new information about all our surroundings at each step. In order to account for this new information, we use $P(k+1)$, the uncertainty update equation.

This factors in both the computed Kalman gain, $K(k+1)$, and the measurement matrix, which is constant. We know that $K(k+1)M < I$ by the following algebra:

$$K(k+1)M = \left[P'(k+1) \cdot M^T \cdot M\right] \cdot \left[MP'(k+1)M^T + C_w\right]^{-1} = \mathbf{1} \cdot [\mathbf{1} - \mathbf{C_w}]^{-\mathbf{1}} < \mathbf{I}$$

Therefore, we are assured that $[0] < \left[I - K(k+1)M\right] < I$, meaning that the uncertainty update equation minimizes the uncertainty equation by a factor dependent

* *oder auf irgendeine Sprache — sondern nicht nur mathisch!*

on the Kalman gain and the measurement matrix.

For simplicity of explanation, we're going to assume that A, B, B', and M are all I, the identity matrix (as is A^T, A's transformation). This assumes that we are moving in ideal conditions — no wind, no ice, no land mines. In reality, each of these three matrices will represent a different function of the environment — maybe A will be friction, M darkness, &c.

Best Guess at Final State:

$$\vec{x}_F(k+1) = \vec{x}(k+1) + K(k+1)\Big(\vec{z}(k+1) - \vec{x}(k+1)\Big)$$

So, check it out:

- As C_v gets really big (i.e., $\rightarrow\ \infty$), that will make the Uncertainty Equation $(P(k+1))$ get really big, too. This means that

$$\lim_{C_v\rightarrow\infty} K(k+1) = \frac{\infty}{\infty} = I$$

  which makes that big ugly $\vec{x}_F(k+1)$:

$$\vec{x}_F(k+1) = \vec{x}(k+1) + I \cdot \Big(\vec{z}(k+1) - \vec{x}(k+1)\Big) = \vec{\mathbf{z}}(\mathbf{k+1}).$$

- Alternatively, as $C_w \rightarrow \infty$, that only affects the denominator of $K(k+1)$ (the (simplified) Kalman gain function):

$$\lim_{C_w\rightarrow\infty} K(k+1) = \frac{1}{\infty} = [0]$$

  therefore,

$$\vec{x}_F(k+1) = \vec{x}(k+1) + [0] \cdot \Big(\vec{z}(k+1) - \vec{x}(k+1)\Big) = \vec{\mathbf{x}}(\mathbf{k+1}).$$

This all just says which parts of what equations to ignore, based on which errors are big — tells you what to trust and what to ignore.